Solution of the Logistic Equation

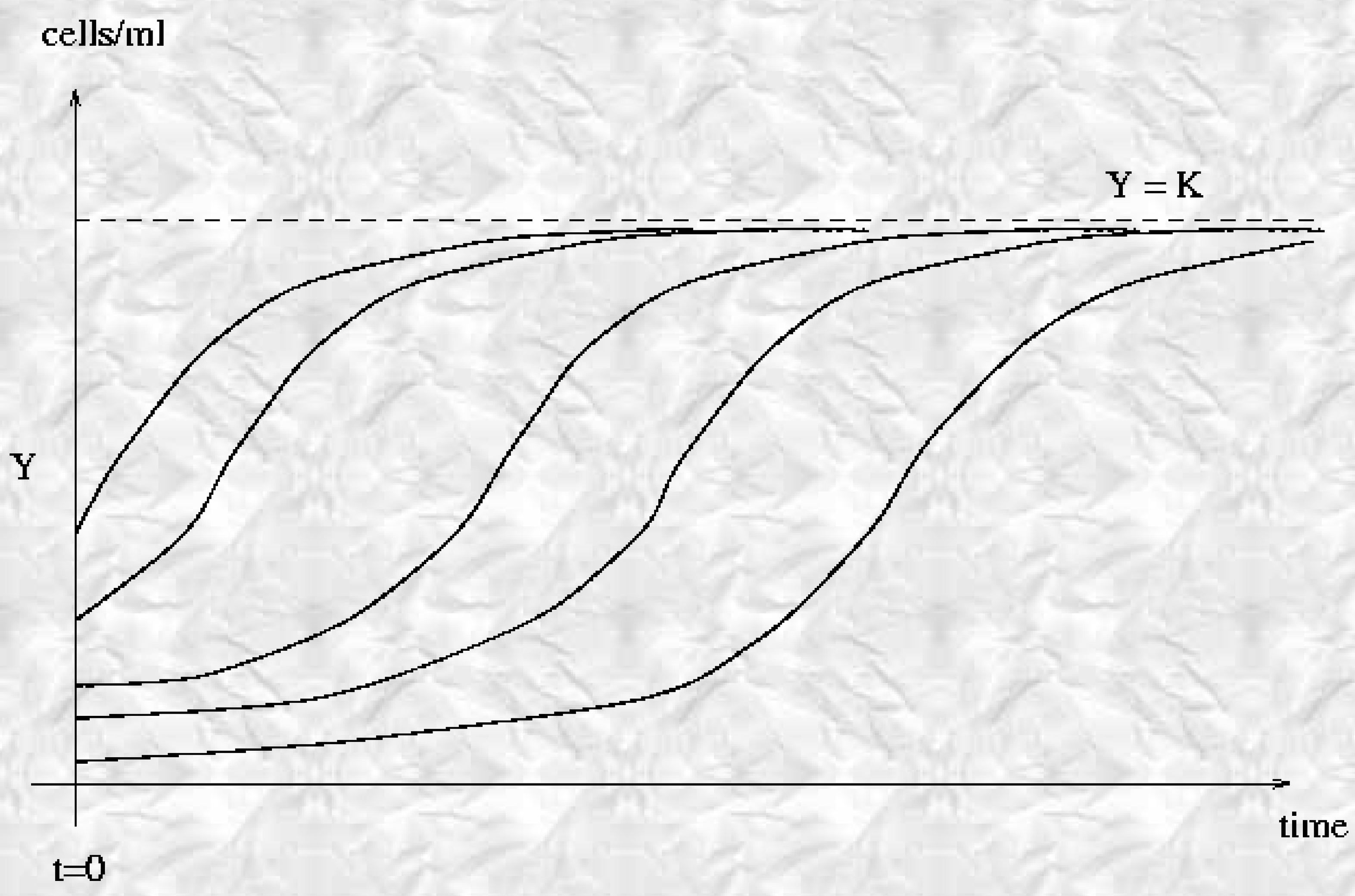

Figure 1: Behavior of typical solutions to the logistic equation. All solutions approach the carrying capacity, K, as time tends to infinity at a rate depending on r, the intrinsic growth rate.

The virtue of having a single, first-order equation representing yeast dynamics is that we can solve this equation using integration techniques from calculus. First we separate variables in (3),

$$\frac{dY}{Y\left(1-\frac{Y}{K}\right)} = r\,dt,$$

annd then we apply partial fractions to the left-hand-side:

$$\frac{dY}{Y} + \frac{\frac{1}{K}dY}{\left(1-\frac{Y}{K}\right)} = \frac{dY}{Y\left(1-\frac{Y}{K}\right)} = r\,dt.$$

Now we can integrate both sides directly, using the facts that

$$\int \frac{dY}{Y} = \ln(Y) + c_1, \quad \frac{1}{K}\int \frac{dY}{\left(1-\frac{Y}{K}\right)} = -\ln\left(1-\frac{Y}{K}\right) + c_2,$$

and

$$\int r\,dt = rt + c_3.$$

Putting these three integrals together, relabelling constants $k = c_3 - c_1 - c_2$, and using $\ln(z) - \ln(w) = \ln(z/w)$ gives

$$\ln\left[\frac{Y}{\left(1-\frac{Y}{K}\right)}\right] = k + rt,$$

or, exponentiating both sides,

$$\frac{Y}{\left(1-\frac{Y}{K}\right)} = e^{k+rt} = Ce^{rt},$$

where $C = e^k$. Note that when $t = 0$ we can see that

$$C = \frac{Y_0}{\left(1-\frac{Y_0}{K}\right)} = \frac{KY_0}{K - Y_0}.$$

Now we can solve for Y,

$$Y = \left(1-\frac{Y}{K}\right)Ce^{rt},$$

$$Y\left(1+\frac{C}{K}e^{rt}\right) = Ce^{rt},$$

$$Y = \frac{Ce^{rt}}{\left(1+\frac{C}{K}e^{rt}\right)}.$$

This is the general form of the solution to the logistic equation, (3). If we want to see explicitly how the initial conditions for the yeast population figure in we can substitute $C = KY_0/(K - Y_0)$ and $K = aS_0 + Y_0$ to get

$$Y = \frac{KY_0e^{rt}}{K + Y_0\left(e^{rt}-1\right)} = \frac{Y_0(aS_0+Y_0)e^{rt}}{aS_0 + Y_0e^{rt}}. \tag{4}$$

The behavior of typical solutions is plotted in Figure 1.

James Powell
2000-07-31