Reduction to a Single First-Order Equation

It is more convenient to solve a single equation instead of both of (1, 2), if possible. Fortunately, since both sides of (1) are derivatives, it is dead easy to integrate this equation. Let the initial conditions be given as $Y(t=0)=Y_0$ and $S(t=0)=S_0$, and integrate both sides of (1):

$$\begin{aligned}
-a\int_0^t \dot{S}\;dt &= \int_0^t \dot{Y}\;dt,\\
-a(S(t)-S(0)) &= Y(t)-Y(0),\\
S(t)-S_0 &= -\frac{1}{a}(Y(t)-Y_0),\\
S(t) &= S_0-\frac{1}{a}(Y(t)-Y_0)=S_0+\frac{Y_0}{a}-\frac{Y(t)}{a}.
\end{aligned}$$

Notice that we have now solved for S in terms of Y and known quantities. Substituting this result into (2) gives

$$\begin{aligned}
\dot{Y} &= bY\left[S_0+\frac{Y_0}{a}-\frac{Y(t)}{a}\right],\\
&= \frac{b}{a}Y\left[aS_0+Y_0-Y\right],\\
&= \frac{b(aS_0+Y_0)}{a}Y\left[1-\frac{Y}{(aS_0+Y_0)}\right].
\end{aligned}$$

Now, if we define

$$K=(aS_0+Y_0) \quad \text{and} \quad r=\frac{b(aS_0+Y_0)}{a}$$

we get the standard ecological form of the logistic equation:

$$\dot{Y}=rY\left(1-\frac{Y}{K}\right), \tag{3}$$

where r is the intrinsic growth rate of this population for this density of sugar and K is the carrying capacity of this sugar solution.

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